The Clown Lottery

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1 Problem

A seemingly trustworthy ex-carnival clown offers you a special lottery ticket. The lottery ticket has 100 slots in which you must place all of the numbers 1-100. The trick is that you don't know what order the numbers should go in, and order matters.

"But wait!" The clown says. "You only need to get one number (or more) in the right position in order to win! And for every winning \$1 ticket, I'll give you \$1.50. That's an easy profit of 50 cents per winning ticket! I'll even spot you one million dollars so you can buy one million tickets (pre-filled completely randomly)!"

Should you take the clown's offer?

More formally: Suppose the numbers $1..N$ are ordered randomly in a list. What are the chances that at least one number is equal to its position in the list?

Example List: $5, 2, 1, 4, 3$ (in this list, 2 and 4 are equal to their position in the list since they are the second and fourth elements in the list).

What does the probability converge to as N approaches infinity?

2 Informal Solution using Probability

This is my slightly hand-wavy solution to the problem. It's also the way that I initially solved it. See section 3 for a more rigorous proof and a better explanation of the details.

2.1 The Probability of a Winning Ticket

Consider tickets with n slots. Let w_k represent the event that a winning ticket has the number k in the correct slot. We want to know what the chances are of any of the events occurring... i.e. the chances of any of the numbers being in the correct slot. So we need to calculate the probability $P(w_1 \cup w_2 \cup w_3 \cup ... \cup w_n)$.

$$
\begin{aligned}
&P(w_1 \cup w_2 \cup w_3 \cup ... \cup w_n) = \\
&\sum_{k=1}^{n} P(w_k) - \sum_{j \neq k} P(w_j \cap w_k) + \sum_{i \neq j \neq k} P(w_i \cap w_j \cap w_k) - ...
\end{aligned} \tag{1}
$$

This is called the inclusion-exclusion principle. To calculate the probability that event 1 or event 2 or event 3, etc... will occur, you add the probabilities of the individual events, then subtract the probabilities that any 2 of them both happen, then add the probabilities that any 3 of them all happen, and so on.*

The probability of a particular number being in the correct slot is 1 in n. This is because given a number, there are n slots which are equally likely for it to be in. The probability of 2 numbers being in the correct slots is $\frac{1}{n} \cdot \frac{1}{n-1}$. This is because there's a 1 in n chance of the first number being in the right slot, and there are only $n-1$ slots for the second number to choose from.

Extending this reasoning we see:

$$
\begin{aligned}
P(w_k) &= \frac{1}{n} \\
P(w_j \cap w_k) &= \frac{1}{n(n-1)} \\
P(w_i \cap w_j \cap w_k) &= \frac{1}{n(n-1)(n-2)} \\
... & \\
P(m \text{ different } w_k \text{ events occurring}) &= \frac{(n-m)!}{n!}
\end{aligned} \tag{2}
$$

There are n different w_k events.
There are $\frac{n(n-1)}{2!}$ different events of the form $w_j \cap w_k$.
In general, there are $\binom{n}{k}$ possible ways to choose k different events from a set of

*Refer to Appendix A for an explanation and proof of the inclusion-exclusion formula.

n events. Combining this fact with equations (1) and (2) gives:

$$P(w_1 \cup w_2 \cup w_3 \cup ... \cup w_n) = \sum_{k=1}^{n} \binom{n}{k} \frac{(n-k)!}{n!} \cdot (-1)^{k-1} \tag{3}$$

Since the choose function $\binom{n}{k} = \frac{n!}{k!(n-k!)}$, we can simplify the above equation:

$$P(w_1 \cup w_2 \cup w_3 \cup ... \cup w_n) = \sum_{k=1}^{n} \frac{1}{k!} \cdot (-1)^{k-1} \tag{4}$$

Let $f(n) = P(w_1 \cup w_2 \cup w_3 \cup ... \cup w_n) =$ the probability of a winning ticket. This give us the result:

$$\boxed{f(n) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + ...} \tag{5}$$

2.2 Relation to Euler's Number

Euler's number $e = 2.718281828...$ can be defined by the infinite series*:

$$e^x = 1 + \frac{x}{1!} + \frac{x}{2!} + \frac{x}{3!} + ... \tag{6}$$

Plugging in $x = -1$ we get

$$\begin{aligned} e^{-1} &= 1 + \frac{-1}{1!} + \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + ... \\ &= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + ... \end{aligned} \tag{7}$$

Notice that the first n fractions of e^{-1} are equal to $-f(n)$. Removing the 1 from the beginning of e^{-1}, we have

$$\begin{aligned} &\text{The first } n \text{ terms of } e^{-1} - 1 = -f(n) \\ &\quad \Rightarrow f(n) = \text{first } n \text{ terms of } 1 - e^{-1} \end{aligned} \tag{8}$$

Therefore

$$\boxed{\lim_{n \to \infty} f(n) = 1 - \frac{1}{e}} \tag{9}$$

2.3 Considering the Clown's Offer

The clown is willing to pay \$1.50 for each winning ticket. Since each ticket costs \$1, this means you win \$0.50 for a winning ticket and lose \$1 for a losing ticket.

*See Appendix B for a derivation of Euler's number and e^x.

Define W as the probability of a winning ticket and L as the probability of a losing ticket. The expected value of a ticket is

$$E = 0.50W - 1L \tag{10}$$

We want to solve for W and L such that $E > 0$. Notice that $W + L = 1$ because the only possibilities are winning and losing. Therefore $L = 1 - W$. Substituting into E we get:

$$E = 0.50W - (1 - W) = \frac{W}{2} - 1 + W = \frac{3W}{2} - 1 \tag{11}$$

$$E > 0 \Rightarrow \frac{3W}{2} - 1 > 0 \Rightarrow W > \frac{2}{3} \approx 0.6667 \tag{12}$$

Therefore the probability of a winning ticket needs to be greater than $\frac{2}{3}$ for the offer to be worth considering.

We need to calculate $f(100)$, the probability of a random ticket with 100 slots being a winning ticket. From equation (9) we know that the probability of a winning ticket $f(n)$ approaches $1 - \frac{1}{e}$ as n (the number of slots) grows.

$$f(n) \approx 1 - \frac{1}{e} \approx 1 - \frac{1}{2.71828} \approx 0.6321 < \frac{2}{3} \tag{13}$$

As n grows larger, the probability converges to a number less than 0.6667. Consider $f(n)$ for various values:

$$\begin{aligned}
f(1) &= \frac{1}{1!} &&= 1\\
f(2) &= \frac{1}{1!} - \frac{1}{2!} &&= 0.5\\
f(3) &= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} &&\approx 0.6667\\
f(4) &= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} &&= 0.625\\
f(5) &= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} &&\approx 0.6333\\
f(6) &= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} &&\approx 0.6319\\
f(7) &= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!} &&\approx 0.6321\\
f(8) &= \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!} - \frac{1}{8!} &&\approx 0.6321
\end{aligned} \tag{14}$$

You can see that the probability converges to 0.6321.... So the expected value of each ticket is negative. Therefore it would be a mistake to accept the seemingly trustworthy ex-carnival clown's offer.

2.4 Consequences of the Clown's Offer

If you did accept the clown's offer, then using equation (11) we see that you would make around $\frac{3}{2} \cdot (0.6321) - 1$ dollars per ticket.

$$\$[\frac{3}{2} \cdot (0.6321) - 1] \approx -\$0.05185 \tag{15}$$

You would on average lose around 5 cents per tickets. If you accepted the offer of 1 million tickets, then you would lose about $51,850.

But thanks to math, you were able to avoid that disastrous outcome!

2.5 Computer Simulation

In the interest of verifying the answer, I wrote a simple Python program to simulate The Clown Lottery scenario:

```
import random

def IsWinning(theList):
    for anIndex, anElem in enumerate(theList):
        if anElem==anIndex:
            return True

aMoney = 0.0
aList = range(100)
for i in xrange(1000000):
    random.shuffle(aList)
    if IsWinning(aList): aMoney += 0.5
    else: aMoney -= 1.0

print aMoney
```

Running the program 5 times produced the following results: -$52,073.50, -$51,589, -$51,178, -$52,891, -$52,676.

2.6 Discussion

I find it counter-intuitive that if you were to pick 100 numbers at random (or 1 million numbers or 1 trillion numbers), the chances are greater than $\frac{1}{3}$ that none of them would be equal to the order that they were picked in. But nobody I've asked has thought it was a good idea to accept the seemingly trustworthy ex-carnival clown's offer. Another win for street smarts!

The original way I've seen this problem described is in terms of letters and envelopes. Imagine you have N envelopes addressed to N different people. If you placed the letters in the envelopes at random, then what are the chances that none of the letters went in the right envelope?

Does it seem intuitive that there's over a $\frac{1}{3}$ chance that *none* of the letters will go in the correct envelope? Even if there were 1 million envelopes?

Whatever the case, I'm certain that everybody can agree that I made the problem more relevant to the age that we live in.

3 Formal Proof

In order to present a more formal proof of the solution, it will be helpful to define the following terms:

n-ticket	= a ticket with n slots containing the numbers $1..n$
correct slot	= a slot which contains a number equal to its position
winning ticket	= a ticket with at least one correct slot
$f(n)$	= the probability of a winning n-ticket
$g(n)$	= the number of unique winning n-tickets
T_n	= the set of all n-tickets
W_n	= the set of all winning n-tickets
S_n^k	= the set of all n-tickets where the k^{th} slot is correct
$\mathbb{S}_n$	= the set of all of the S_n^k sets
k-intersection	= the intersection of k different sets
$\mathbb{C}_n^k$	= the set of all possible k-intersections of sets in $\mathbb{S}_n$

3.1 Explanation of T_n and W_n

The T_n set represents the set of all possible tickets with n slots. This is not to be confused with the 1 million tickets the clown is offering you. T_n contains every possible unique ticket you could make with n numbers.

To construct W_n, you'd go through all of the tickets in T_n and set aside any that had one or more numbers in the correct slot. These would be the tickets that should go in W_n.

3.2 Explanation of S_n^k

To get a better idea of what the S_n^k sets represent, let's consider S_{100}^{42}. This is the set of all lottery tickets with 100 slots that have the number 42 in the 42^{nd} slot. If you wanted to make this set in real life, then you'd go through all of the tickets in T_{100} and set aside the ones that had 42 in the correct slot.

Now let's consider S^7_{100}. You'd make this set in a similar way to S^{42}_{100}. Go through all of the tickets in T_{100} and put aside the ones with 7 in the correct slot.

Here's an important question. Can S^7_{100} and S^{42}_{100} contain some of the same tickets? Yes! They contain many of the same tickets. There are tons of tickets that have both 7 and 42 in the correct slots.

3.3 Explanation of $\mathbb{S}_n$

Here's where things start to get a little dicey – introducing sets that contain other sets! All I can do is apologize and then try to explain.

$\mathbb{S}_n$ is a set that contains sets as its elements. Think of it as a box containing a bunch of boxes which each contain lottery tickets.

Let's look at how you'd make $\mathbb{S}_{10}$ in real life.

First you'd create 10 copies of T_{10}. From the first copy you'd construct S^1_{10} by finding all of the tickets with the number 1 in the first slot. Then you'd go through the next copy and construct S^2_{10} by finding all of the tickets with the number 2 in the second slot. And you'd continue to do this in order to construct the other sets with the 3-10 in the correct slots.

Now you have 10 sets, each one containing all the possible tickets that have a particular number in the correct slot. Imagine you've put all the tickets for each set in a different box. Now place those 10 boxes in a bigger box. That bigger box is the $\mathbb{S}_{10}$ set.

3.4 Explanation of k-intersections

Ok. What the heck is a k-interection? This is probably the worst of all my definitions.

First let me explain what a set intersection is. The intersection of two sets A and B is another set containing the elements that A and B have in common. This intersection is written as $A \cap B$.

Let's consider the example from above of S^7_{100} and S^{42}_{100}.

What is $S^7_{100} \cap S^{42}_{100}$?

It's a set containing all of the lottery tickets that have both 7 and 42 in the correct slots.

Now what is a k-intersection? It's a set produced from intersecting k different sets. Call the sets $A_1, A_2, A_3, ..., A_k$. The new set is $A_1 \cap A_2 \cap A_3 \cap ... \cap A_k$. It contains only the elements that $A_1, A_2, A_3, ..., A_k$ all have in common.

What would we call $S_{100}^7 \cap S_{100}^{42}$? It's a 2-intersection. It only contains lottery tickets with a 7 in the seventh slot and a 42 in the 42^{nd} slot.

3.5 Explanation of $\mathbb{C}_n^k$

$\mathbb{C}_n^k$ is tied for the worst of my definitions. Unfortunately, the whole solution hinges on this concept.

$\mathbb{C}_n^k$ is another set of sets like $\mathbb{S}_n$.

Let's look at the example of $\mathbb{C}_5^3$. The definition says that $\mathbb{C}_5^3$ is the set of all possible 3-intersections of sets in $\mathbb{S}_5$

The way you would construct $\mathbb{C}_5^3$ is you'd take every combination of groups of 3 sets from $\mathbb{S}_5$ and you'd intersect them. Each of these 3-intersections would be a set in $\mathbb{C}_5^3$.

Specifically $\mathbb{C}_5^3$ would contain the following 10 sets:

$$S_5^1 \cap S_5^2 \cap S_5^3, \quad S_5^1 \cap S_5^2 \cap S_5^4, \quad S_5^1 \cap S_5^2 \cap S_5^5$$
$$S_5^1 \cap S_5^3 \cap S_5^4, \quad S_5^1 \cap S_5^3 \cap S_5^5, \quad S_5^1 \cap S_5^4 \cap S_5^5$$
$$S_5^2 \cap S_5^3 \cap S_5^4, \quad S_5^2 \cap S_5^3 \cap S_5^5, \quad S_5^2 \cap S_5^4 \cap S_5^5$$
$$S_5^3 \cap S_5^4 \cap S_5^5$$

Think of $\mathbb{C}_5^3$ as a box that contains 10 boxes of lottery tickets. One box has all the tickets where slots 1,2, and 3 are correct. One box has the tickets where slots 1,2, and 4 correct. And so forth.

It's important to realize that the same ticket can be contained in multiple boxes. For instance the ticket {1,2,3,4,5} is in every one of the above 10 boxes.

3.6 Conclusion

Finally, we can start solving the problem! The key to solving a math problem is often in coming up with good definitions.

4 Solution Using Set Theory

Notice that the probability of receiving a winning ticket is simply the number of winning tickets divided by the total number of tickets.*

$$f(n) = \frac{g(n)}{|T_n|} \tag{16}$$

The number of n-tickets is n factorial.†

$$|T_n| = n! \tag{17}$$

The number of winning n-tickets is

$$g(n) = |W_n| \tag{18}$$

Notice that W_n is just the union‡ of all the sets of n-tickets which have either the 1st slot correct, the 2nd slot correct, the 3rd slot correct, etc...

$$W_n = \bigcup_{k=1}^{n} S_n^k \tag{19}$$

Now here is the key to the entire problem. We want to calculate the size of W_n. How many possible winning tickets are there?

$$|W_n| = |\bigcup_{k=1}^{n} S_n^k| \tag{20}$$

Rewriting this in terms of $\mathbb{S}_n$ gives

$$|W_n| = |\bigcup_{A \in \mathbb{S}_n} A| \tag{21}$$

How do we calculate the size of a union of sets? The inclusion-exclusion principle gives us a formula for determinig the size of a union of sets. Refer to Appendix A for an explanation and proof of the formula.

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*$|B|$ denotes the number of elements contained in the set B.

†To see why the number of tickets is $n!$, imagine how you could construct all of the tickets. For the first slot there would be n possibilities. For each of those n starting possibilities, there would be $n-1$ remaining possibilities since the numbers aren't allowed to repeat. For each of those $n \cdot (n-1)$ tickets with the first two numbers selected, there would be $n-2$ remaining numbers to choose from. This would give $n(n-1)(n-2)$ possible tickets with the first three numbers filled in. Extending this process to fill in all of the slots you find that there are $n(n-1)(n-2)...3 \cdot 2 \cdot 1$ possible tickets which is n factorial.

‡The union of two sets $A \cup B$ is a set which contains all of the elements in A and B. Note that sets never contain duplicate elements. They can only contain at most one copy of each distinct item. In this case, lottery tickets.

Applying the inclusion-exclusion principle to equation (21) gives

$$|\bigcup_{A\in\mathbb{S}_n} A| = \sum_{A\in\mathbb{S}_n}|A| - \sum_{A\in\mathbb{C}_n^2}|A| + \sum_{A\in\mathbb{C}_n^3}|A| - \sum_{A\in\mathbb{C}_n^4}|A| + ... \tag{22}$$

Notice that $\mathbb{S}_n = \mathbb{C}_n^1$. This is because the set of 1-intersections from $\mathbb{S}_n$ is simply all of the S_n^k sets. So we can rewrite the equation

$$|\bigcup_{A\in\mathbb{S}_n} A| = \sum_{k=1}^{n}\sum_{A\in\mathbb{C}_n^k}|A|\cdot(-1)^{k-1} \tag{23}$$

Each set in $\mathbb{C}_n^k$ contains tickets where k specific slots are correct. The number of tickets with k specific correct slots is $(n-k)!$ *

$$\forall A \in \mathbb{C}_n^k : |A| = (n-k)! \tag{24}$$

The number of sets in each $\mathbb{C}_n^k$ is $\binom{n}{k}$. † This is because $\mathbb{C}_n^k$ consists of all of the possible k-intersections from $\mathbb{S}_n$. To construct all of the k-intersections, you must choose every combination of k sets from the n sets contained in $\mathbb{S}_n$.

$$\forall k : |\mathbb{C}_n^k| = \binom{n}{k} \tag{25}$$

Using these two results and the fact that $g(n) = |\bigcup_{A\in\mathbb{S}_n} A|$ ‡ we can rewrite equation (23) as follows

$$g(n) = \sum_{k=1}^{n}\binom{n}{k}(n-k)!\cdot(-1)^{k-1} \tag{26}$$

Substituting the formula for $\binom{n}{k}$ we get

$$g(n) = \sum_{k=1}^{n}\frac{n!}{k!(n-k)!}(n-k)!\cdot(-1)^{k-1} \tag{27}$$

Cancelling out the $(n-k)!$ gives

$$g(n) = \sum_{k=1}^{n}\frac{n!}{k!}\cdot(-1)^{k-1} \tag{28}$$

*Once you choose the k correct slots, there are $(n-k)$ slots left which can be arranged in $(n-k)!$ ways.

†The choose function $\binom{n}{k}$ specifies how many different sets of k elements can made from a bigger set of n elements. $\binom{n}{k} = \frac{n(n-1)(n-2)...(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$
To see why this is true, consider picking the first element. There are n choices. For the second element there are $n-1$ choices. Keep picking until you've picked k elements. Since you're constructing a set, the order in which you picked the elements doesn't matter. There are $k!$ ways to represent a list of k elements. Therefore divide by $k!$ to remove the duplicates.

‡see equations (18) and (21)

Factoring the $n!$ out of the sum gives

$$\boxed{g(n) = n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}} \tag{29}$$

Remember that $f(n) = \frac{g(n)}{|T_n|} = \frac{g(n)}{n!}$. Substituting $g(n)$ into this equation and cancelling the $n!$ gives

$$\boxed{f(n) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}} \tag{30}$$

5 Recursive Solution

Here's a completely different way to solve the problem using a recursive definition of $g(n)$.

5.1 Reasoning about the first slot

Recall that $g(n) =$ the number of unique winning n-tickets.

Let's think of how we'd calculate $g(n)$ by imagining how many winning tickets begin with the numbers $1, 2, 3, ..., n$.

There are $(n-1)!$ winning n-tickets that begin with a 1. That's because the 1 is in the correct slot, so you can arrange the other $n-1$ numbers however you want.

How many winning tickets begin with a 2? Consider the function:

$$\boxed{\text{h(n) = number of winning } n\text{-tickets where one of the numbers is never correct}}$$

When the 2 is in the first slot, it can never cause a winning ticket. So we want to know how many winning tickets there are with the remaining $n-1$ numbers. Think of those remaining numbers as forming an $(n-1)$-ticket. The 1 is one of those numbers, and it can never go in its proper slot. Therefore those remaining numbers can form $h(n-1)$ winning tickets. So the number of winning n-tickets with a 2 in the first slot is $h(n-1)$.

This reasoning holds when any of the numbers $2, 3, 4, ...n$ is in the first slot. Therefore:

$$\begin{aligned}
&\text{The number of winning n-tickets that begin with a 2} &&= h(n-1)\\
&\text{The number of winning n-tickets that begin with a 3} &&= h(n-1)\\
&\text{The number of winning n-tickets that begin with a 4} &&= h(n-1)\\
&...\\
&\text{The number of winning n-tickets that begin with an } n &&= h(n-1)
\end{aligned}$$

Now we can write $g(n)$ in terms of $h(n)$ because we know how many winning tickets start with a $1, 2, 3, 4, ...n$. This represents all of the winning tickets.

$$\boxed{g(n) = (n-1)! + (n-1)h(n-1)} \tag{31}$$

5.2 Figuring out h(n)

How do we calculate $h(n)$?

Think about the tickets we're considering. We're considering tickets with n slots but only $n-1$ numbers that can be correct. Imagine that the tickets have the numbers $2..n$ and the number 0. The zero doesn't belong in any of the slots. It doesn't ever cause a winning ticket.

There are two cases we need to think about:

- The 0 is in the first slot.
- The 0 is in any other slot.

When the 0 is in the first slot, you can think of the ticket as an $(n-1)$-ticket. The 0 doesn't contribute to any wins, but all of the other numbers can fit properly. Therefore there are $g(n-1)$ winning tickets with a 0 in the first slot.

When the 0 isn't in the first slot, then there's either a $2, 3, 4, ...,$ or n in the first slot. In each of these cases, we have an incorrect number in front and $n-1$ numbers remaining. One of those numbers is the 0 which doesn't have a matching slot. Therefore there are $h(n-1)$ winning tickets with a 2 in front. And there are $h(n-1)$ winning tickets with a 3 in front. This holds for all $(n-1)$ leading numbers $2, 3, 4, ..., n$

Using these two cases, we see that

$$\boxed{h(n) = g(n-1) + (n-1)h(n-1)} \tag{32}$$

5.3 Getting rid of h(n)

We've computed $g(n)$ and $h(n)$ as follows:

- $g(n) = (n-1)! + (n-1)h(n-1)$
- $h(n) = g(n-1) + (n-1)h(n-1)$

Let's see if we can manipulate these two equations in order to eliminate the $h(n)$.

Notice that the second term, $(n-1)h(n-1)$, is the same in both equations. Therefore to get rid of it, subtract one from the other.

$$h(n) - g(n) = g(n-1) - (n-1)! \tag{33}$$

Add $g(n)$ to both sides to get

$$h(n) = g(n) + g(n-1) - (n-1)! \tag{34}$$

Now plug this formula for $h(n)$ into equation 31 for $g(n)$

$$\begin{aligned} g(n) &= (n-1)! + (n-1)h(n-1) \\ &= (n-1)! + (n-1)[g(n-1) + g(n-2) - (n-2)!] \\ &\textbf{Distribute the (n-1)} \\ &= (n-1)! + (n-1)g(n-1) + (n-1)g(n-2) - (n-1)(n-2)! \\ &\textbf{Combine the (n-1)(n-2)!} \\ &= (n-1)! + (n-1)g(n-1) + (n-1)g(n-2) - (n-1)! \\ &\textbf{Cancel the two (n-1)! terms} \\ &= (n-1)g(n-1) + (n-1)g(n-2) \end{aligned}$$

Factor out the $(n-1)$ to arrive at this nice equation:

$$\boxed{g(n) = (n-1)\Big[g(n-1) + g(n-2)\Big]} \tag{35}$$

5.4 Finding a non-recursive g(n) - Part 1

Now we have a nice Fibonacci-like definition of $g(n)$. Isn't that what we wanted? Not exactly. We wanted to find a recursive function to avoid all of the pain of reasoning about sets of sets and inclusion-exclusion principles. We still need to convert this into an expression that lets us easily calculate the probability function $f(n)$. That's the actual problem we're trying to solve: What is the probability of getting a winning ticket?

Let's investigate what happens when we plug in the recursive definition of $g(n)$ into itself. This is a bit of tricky step so I'm going to box important parts to help see a pattern.

$$
\begin{aligned}
g(n) &= (n-1)[g(n-1) + g(n-2)] \\
&\quad \textbf{Distribute the (n-1)} \\
&\quad = ng(n-1) \boxed{-g(n-1) + (n-1)g(n-2)} \\
&\quad \textbf{Expand g(n-1) using the recursive formula} \\
&\quad = ng(n-1) - (n-2)[g(n-2) + g(n-3)] + (n-1)g(n-2) \\
&\quad \textbf{Distribute the (n-2)} \\
&\quad = ng(n-1) - (n-2)g(n-2) - (n-2)g(n-3) + (n-1)g(n-2) \\
&\quad \textbf{Combine the g(n-2) terms} \\
&\quad = ng(n-1) \boxed{+g(n-2) - (n-2)g(n-3)} \\
&\quad \textbf{Expand g(n-2) using the recursive formula} \\
&\quad = ng(n-1) + (n-3)[g(n-3) + g(n-4)] - (n-2)g(n-3) \\
&\quad \textbf{Distribute the (n-3)} \\
&\quad = ng(n-1) + (n-3)g(n-3) + (n-3)g(n-4) - (n-2)g(n-3) \\
&\quad \textbf{Combine the g(n-3) terms} \\
&\quad = ng(n-1) \boxed{-g(n-3) + (n-3)g(n-4)}
\end{aligned}
$$

Notice that the boxed parts are all of the form

$$\boxed{(-1)^a \cdot \big[g(n-a) - (n-a)g(n-a-1)\big]} \tag{36}$$

Let's investigate this expression.

$$\text{Let } k(n) = \big[g(n) - ng(n-1)\big] \tag{37}$$

$k(n)$ is just equation (36) with $a = 0$. Let's investigate expanding the g(n) term.

$$
\begin{aligned}
k(n) &= [g(n) - ng(n-1)] \\
&\quad \textbf{Expand g(n) using the recursive formula} \\
&\quad = (n-1)g(n-1) + (n-1)g(n-2) - ng(n-1) \\
&\quad \textbf{Combine the g(n-1) terms} \\
&\quad = -g(n-1) + (n-1)g(n-2) \\
&\quad \textbf{Factor out the minus sign} \\
&\quad = -[g(n-1) - (n-1)g(n-2)]
\end{aligned}
$$

Therefore we see that

$$k(n) = -k(n-1) \tag{38}$$

$k(n-a)$ simply flips signs based on the value of a

$$\Rightarrow k(n) = (-1)^a \cdot k(n-a) \tag{39}$$
$$\Rightarrow k(n) = (-1)^{n-2} \cdot k(n-(n-2)) \tag{40}$$
$$\Rightarrow k(n) = (-1)^n \cdot k(2) \tag{41}$$

Let's calculate $k(2)$. We can use $k(2)$ to find the values for all $k(n)$. In this calcualtion keep in mind that $g(2) = 1$ and $g(1) = 1$. You can calculate these result using the recursive $g(n)$ formula, but you can also just realize that there's 1 winning ticket with 1 slot, and there's 1 winning ticket with 2 slots.

Plugging 2 into equation (37) gives

$$k(2) = g(2) - 2g(1) \tag{42}$$
$$= 1 - 2 \cdot 1 \tag{43}$$
$$= -1 \tag{44}$$

Plugging this value of $k(2) = -1$ back into equation (41) we get

$$k(n) = (-1)^n \cdot k(2) \tag{45}$$
$$= (-1)^n \cdot (-1) \tag{46}$$
$$= (-1)^{n+1} \tag{47}$$

Finally, substitute $k(n) = (-1)^{n+1}$ into equation (37)

$$k(n) = g(n) - ng(n-1) \tag{48}$$
$$\Rightarrow (-1)^{n+1} = g(n) - ng(n-1) \tag{49}$$
$$\tag{50}$$

This leads us to another recursive formula for $g(n)$, but one that only contains $g(n-1)$.

$$\boxed{g(n) = ng(n-1) + (-1)^{n+1}} \tag{51}$$

Just like the equation 35 version of $g(n)$ looks similar to the Fibonacci function, this equation 51 version of $g(n)$ looks very similar to the recursive definition of the factorial function.*

*I just thought I'd mention that. Not for any particular reason really.

5.5 Digression - Other ways of finding g(n)

Without going through all of the work of this proof, you could probably guess equation (51) by investigating small values of $g(n)$. Consider:

$$\begin{aligned} g(1) &= 1 \\ g(2) &= 1 \\ g(3) &= 4 \\ g(4) &= 15 \\ g(5) &= 76 \end{aligned}$$

Assume $g(n) \approx n \cdot g(n-1)$. We see that

$$\begin{aligned} g(1) &= 1 & & \\ g(2) &= 1 & &= 2 \cdot 1 - 1 \\ g(3) &= 4 & &= 3 \cdot 1 + 1 \\ g(4) &= 15 & &= 4 \cdot 4 - 1 \\ g(5) &= 76 & &= 5 \cdot 15 + 1 \end{aligned}$$

Of course hindsight is 20-20. And guessing the equation wouldn't lead to a proof unless you could figure out a way of reasoning out why equation (51) is true. Despite its simplicity, I haven't seen a way to explain it or prove it except by this proof or deriving it from equations (29) or (30) from the proof in section 4.

To be honest, this was the first way I found the recursive $g(n)$ formula. I used equation (30) to derive $g(n)$ as follows:

$$\begin{aligned} & f(n) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots \\ \Rightarrow & f(n) = f(n-1) + \frac{1}{n!} \cdot (-1)^{n-1} \\ & \textbf{Multiply by n!} \\ \Rightarrow & n! \cdot f(n) = n! \cdot f(n-1) + \frac{n!}{n!} \cdot (-1)^{n-1} \\ & \textbf{Note that n! } \cdot \textbf{ f(n) = g(n)} \\ \Rightarrow & g(n) = n! \cdot f(n-1) + 1 \cdot (-1)^{n-1} \\ \Rightarrow & g(n) = n(n-1)! \cdot f(n-1) + 1 \cdot (-1)^{n-1} \\ \Rightarrow & g(n) = ng(n-1) + (-1)^{n-1} \end{aligned}$$

5.6 Finding a non-recursive g(n) - Part 2

Imagine we didn't know the non-recursive version of $g(n)$. How would we go about finding it? We could look at a particular example of $g(n)$ and try to spot a pattern. Consider $g(5)$

$$\begin{aligned}
g(5) &= 5\cdot g(4)+1\\
&= 5\cdot(4\cdot g(3)-1)+1\\
&= 5\cdot(4\cdot(3\cdot g(2)+1)-1)+1\\
&= 5\cdot(4\cdot(3\cdot(2\cdot g(1)-1)+1)-1)+1\\
&= 5\cdot(4\cdot(3\cdot(2\cdot 1-1)+1)-1)+1\\
&= 5\cdot 4\cdot 3\cdot 2-5\cdot 4\cdot 3+5\cdot 4-5+1\\
&= \frac{5!}{1!}-\frac{5!}{2!}+\frac{5!}{3!}-\frac{5!}{4!}+\frac{5!}{5!}
\end{aligned}$$

This suggests the following formula for $g(n)$.

$$\text{Hypothesis: } g(n) = n!\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!} \tag{52}$$

Let's prove this by induction.

Base case $n = 1$:

$$g(1) = 1!\frac{1}{1!} = 1 \tag{53}$$

So the base case is true. Now assume that the hypothesis (equation 52) holds for n. We want to verify that it holds for $n+1$.

$$g(n+1) = (n+1)g(n) + (-1)^n$$

swapping the two terms and plugging in equation (52) for g(n)

$$= (-1)^n + (n+1)n!\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}$$

Combining (n+1)n!

$$= (-1)^n + (n+1)!\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}$$

Multiplying by $\boldsymbol{\frac{(n+1)!}{(n+1)!}}$

$$= (-1)^n\cdot\frac{(n+1)!}{(n+1)!} + (n+1)!\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}$$

Rearranging the $\boldsymbol{(-1)^n}$ and $\boldsymbol{(n+1)!}$

$$= (n+1)!\cdot\frac{(-1)^n}{(n+1)!} + (n+1)!\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k!}$$

Adding the n+1 term to the sum

$$= (n+1)!\sum_{k=1}^{n+1}\frac{(-1)^{k-1}}{k!}$$

So the hypothesis holds for $n+1$. Therefore, by induction, the hypothesis holds for all n.

Now that we have $g(n)$, we can find $f(n)$. Since there are $n!$ possible n-tickets, the probability of a winning ticket is

$$f(n) = \frac{g(n)}{n!} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \tag{54}$$

5.7 Discussion

There you have it. We managed to prove the problem without having to count sets or reason about probabilities (beyond some simple factorial logic). Was it easier than the first rigorous proof? I don't think so. While it didn't have the confusing idea of sets of sets and inclusion-exclusion, it did have a lot more algebraic manipulation as well as inpsection of recursive formulas searching for ways to transform them.

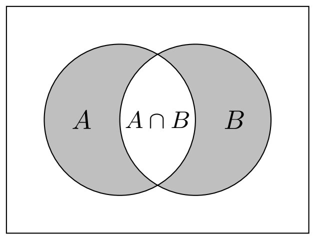
$g(n) = (n-1)! + (n-1)h(n-1)$	From two recursive functions
$h(n) = g(n-1) + (n-1)h(n-1)$	referring to each other.
$g(n) = (n-1)[g(n-1) + g(n-2)]$	To one function with two $g(n)$'s.
$g(n) = ng(n-1) + (-1)^{n-1}$	To one with one $g(n)$.
$g(n) = n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}$	And finally the non-recursive version.

And I'm not sure that I would have been able to do it if I didn't know the answer I was trying to find ahead of time.

Appendix A The Inclusion-Exclusion Principle

A.1 The Union of 2 Sets

If we have 2 sets A and B, how do we calculate the size of the union of those two sets? Do we just add the sizes of each set?



$$|A \cup B| = |A| + |B| \text{ ???}$$

No. If we add the size of the sets then we end up double-counting the elements that they have in common. We have to subtract the number of common elements to get the right size. The common elements are represented by the intersection of A and B: $A \cap B$ which for brevity I'll refer to as AB.

$$|A \cup B| = |A| + |B| - |AB| \tag{55}$$

A.2 The Union of 3 Sets

The method of inclusion-exclusion demonstrated in equation (55) can be extended to the union of 3 sets as follows.

Consider $|A \cup B \cup C|$

$|A \cup B \cup C| =$
$|(A \cup B) \cup C| =$ Apply equation (55) to $|(A \cup B) \cup C|$
$|A \cup B| + |C| - |(A \cup B)C| =$ Apply equation (55) to $|A \cup B|$
$|A| + |B| - |AB| + |C| - |(A \cup B)C| =$ Distribute the C in $|(A \cup B)C|$
$|A| + |B| - |AB| + |C| - |AC \cup BC| =$ Apply equation (55) to $|AC \cup BC|$
$|A| + |B| - |AB| + |C| - (|AC| + |BC| - |ACBC|) =$
$|A| + |B| - |AB| + |C| - |AC| - |BC| + |ACBC|$

Thus we see:

$$|A \cup B \cup C| = |A| + |B| + |C| - |AB| - |AC| - |BC| + |ABC| \tag{56}$$

This suggests a pattern.

A.3 The Union of N Sets

Consider the sets $A_1, A_2, A_3, ...$
Let $\mathbb{A}_n$ = the set of sets containing $A_1, A_2, A_3, ..., A_n$.
Let $\mathbb{B}_n^k$ = the set of all possible k-intersections of sets from $\mathbb{A}_n$. [*] [†]

Hypothesis:

$$|\bigcup_{k=1}^{n} A_k| = \sum_{C\in\mathbb{B}_n^1} |C| - \sum_{C\in\mathbb{B}_n^2} |C| + \sum_{C\in\mathbb{B}_n^3} |C| - \sum_{C\in\mathbb{B}_n^4} |C| + ... \tag{57}$$

Combining the sums together:

$$|\bigcup_{k=1}^{n} A_k| = \sum_{k=1}^{n} \sum_{C\in\mathbb{B}_n^k} |C| \cdot (-1)^{k-1} \tag{58}$$

Let's prove this hypothesis by induction.

Base case $n = 2$:

$$|\bigcup_{k=1}^{2} A_k| = \sum_{k=1}^{2} \sum_{C\in\mathbb{B}_2^k} |C| \cdot (-1)^{k-1} \tag{59}$$

Expanding the union on the left and the sum on the right:

$$|A_1 \cup A_2| = \sum_{C\in\mathbb{B}_2^1} |C| - \sum_{C\in\mathbb{B}_2^2} |C| \tag{60}$$

Expanding the sums on the right:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 A_2| \tag{61}$$

Therefore the base case is true.
Now assume that the hypothesis (equation 58) holds for n sets $A_1..A_n$.
We want to verify that it holds for $n+1$ sets.

$$\text{Let } D_n = \bigcup_{k=1}^{n} A_k \tag{62}$$

$$|D_n| = \sum_{k=1}^{n} \sum_{C\in\mathbb{B}_n^k} |C| \cdot (-1)^{k-1} \tag{63}$$

---

[*] See Section 3 for an explanation of k-intersections.

[†] Technically, $\mathbb{B}_n^k$ is a multiset. A multiset is a set that can contain the same elements multiple times. What's important is that the k-intersections contained in $\mathbb{B}_n^k$ never intersect the exact same group of sets from $\mathbb{A}_n$ even if some of the intersections give the same result.

Let's look what happens when we add A_{n+1} to the mix:

$$|D_n \cup A_{n+1}| = |D_n| + |A_{n+1}| - |D_n \cap A_{n+1}| \tag{64}$$

Expanding D_n we find

$$|D_n \cap A_{n+1}| = |\bigcup_{k=1}^{n} A_k A_{n+1}| \tag{65}$$

The right side of the equation is the size of the union of n sets. Therefore we can use the induction hypothesis (equation 58) to see that

$$|\bigcup_{k=1}^{n} A_k A_{n+1}| = \sum_{k=1}^{n} \sum_{C \in \mathbb{B}_n^k} |C \cap A_{n+1}| \cdot (-1)^{k-1} \tag{66}$$

Define $\mathbb{B}*_n^k$ as the set* of k-intersections of $\mathbb{B}_n^k$ that include A_n
Substituting $\mathbb{B}*_{n+1}^k$ into the previous equation gives

$$|\bigcup_{k=1}^{n} A_k A_{n+1}| = \sum_{k=2}^{n+1} \sum_{C \in \mathbb{B}*_{n+1}^k} |C| \cdot (-1)^{k} \tag{67}$$

Now using equations (63), (64), and (67)

$$|D_n \cup A_{n+1}| = \sum_{k=1}^{n} \sum_{C \in \mathbb{B}_n^k} |C| \cdot (-1)^{k-1} + |A_{n+1}| - \sum_{k=2}^{n+1} \sum_{C \in \mathbb{B}*_{n+1}^k} |C| \cdot (-1)^{k} \tag{68}$$

Distributing the minus on the right gives

$$|D_n \cup A_{n+1}| = \sum_{k=1}^{n} \sum_{C \in \mathbb{B}_n^k} |C| \cdot (-1)^{k-1} + |A_{n+1}| + \sum_{k=2}^{n+1} \sum_{C \in \mathbb{B}*_{n+1}^k} |C| \cdot (-1)^{k-1} \tag{69}$$

Notice that A_{n+1} is the only set in $\mathbb{B}*_{n+1}^1$.
So we can add A_{n+1} to the sum on the right.

$$|D_n \cup A_{n+1}| = \sum_{k=1}^{n} \sum_{C \in \mathbb{B}_n^k} |C| \cdot (-1)^{k-1} + \sum_{k=1}^{n+1} \sum_{C \in \mathbb{B}*_{n+1}^k} |C| \cdot (-1)^{k-1} \tag{70}$$

$\mathbb{B}_n^k \cap \mathbb{B}*_{n+1}^k = \emptyset$ because $\mathbb{B}*_{n+1}^k$ always has A_{n+1} in its intersections and $\mathbb{B}_n^k$ never does.

$\mathbb{B}_n^k \cup \mathbb{B}*_{n+1}^k = \mathbb{B}_{n+1}^k$ because $\mathbb{B}*_{n+1}^k$ contains the missing A_{n+1} intersections.

*$\mathbb{B}*_n^k$ is technically a multiset like $\mathbb{B}_n^k$

Using the above 2 facts and combining the sums in (equation 70) gives

$$|D_n \cup A_{n+1}| = \sum_{k=1}^{n+1} \sum_{C \in \mathbb{B}_{n+1}^k} |C| \cdot (-1)^{k-1} \tag{71}$$

Expanding D_n gives

$$|\bigcup_{k=1}^{n+1} A_k| = \sum_{k=1}^{n+1} \sum_{C \in \mathbb{B}_{n+1}^k} |C| \cdot (-1)^{k-1} \tag{72}$$

So the hypothesis holds for $n + 1$.

Therefore, by induction, the hypothesis holds for all n. Inclusion-Exclusion Formula:

$$\boxed{|\bigcup_{k=1}^{n} A_k| = \sum_{C \in \mathbb{B}_n^1} |C| - \sum_{C \in \mathbb{B}_n^2} |C| + \sum_{C \in \mathbb{B}_n^3} |C| - \sum_{C \in \mathbb{B}_n^4} |C| + ...} \tag{73}$$

A.4 Adding Probabilities

The Inclusion-Exclusion principle can be used to help calculate probabilities. This is because a probability can be defined in terms of sets. A probability event is simply a subset of all possible outcomes under consideration.

Let $S =$ the set of all outcomes
Let $A =$ a subset of S

The probability that A occurs is

$$P(A) = \frac{|A|}{|S|}$$

The probability is the number of possibilities that result in A, divided by the total number of possibilities.

Let's look at a specific example. Suppose we want to evaluate the probability of rolling an even number on a 6-sided die. In this case

$$S = \{1, 2, 3, 4, 5, 6\}$$
$$A = \{2, 4, 6\}$$

Therefore the probability that we roll an even number is

$$P(A) = \frac{|A|}{|S|} = \frac{3}{6} = \frac{1}{2} \tag{74}$$

What if we wanted to calculate the probability that we roll an even number or that we roll a number less than or equal to 3? In this case we have:

S = {1,2,3,4,5,6}	all possibilities
A = {2,4,6}	even possibilities
B = {1,2,3}	possibilities ≤ 3

So the probability of rolling an even number or a number ≤ 3 is

$$P(A \cup B) = \frac{|A \cup B|}{|S|} \tag{75}$$

Calculate $|A \cup B|$ using the inclusion-exclusion principle:

$$\begin{aligned} AB &= \{2,4,6\} \cap \{1,2,3\} = \{2\} \\ |A| &= |\{2,4,6\}| = 3 \\ |B| &= |\{1,2,3\}| = 3 \\ |AB| &= |\{2\}| = 1 \\ |A \cup B| &= |A| + |B| - |AB| \\ |A \cup B| &= 3 + 3 - 1 = 5 \end{aligned}$$

Therefore the probablity that we roll an even number or that we roll a number less than or equal to 3 is

$$P(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{5}{6} \tag{76}$$

In general,

$$P(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B| - |AB|}{|S|} = \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|AB|}{|S|} \tag{77}$$

Therefore

$$\boxed{P(A \cup B) = P(A) + P(B) - P(AB)} \tag{78}$$

For n events we can use the full inclusion-exclusion principle to see that

$$\begin{aligned} &P(A_1 \cup A_2 \cup A_3 \cup ... \cup A_n) = \\ &\frac{|A_1 \cup A_2 \cup A_3 \cup ... \cup A_n|}{|S|} = && \text{(Using Equation 73)} \\ &\frac{\sum_{C \in \mathbb{B}_n^1} |C| - \sum_{C \in \mathbb{B}_n^2} |C| + \sum_{C \in \mathbb{B}_n^3} |C| - \sum_{C \in \mathbb{B}_n^4} |C| + ...}{|S|} = \\ &\sum_{C \in \mathbb{B}_n^1} \frac{|C|}{|S|} - \sum_{C \in \mathbb{B}_n^2} \frac{|C|}{|S|} + \sum_{C \in \mathbb{B}_n^3} \frac{|C|}{|S|} - \sum_{C \in \mathbb{B}_n^4} \frac{|C|}{|S|} + ... \end{aligned}$$

Because the sets in $\mathbb{B}_n^k$ are the k-intersections of $\{A_1, A_2, A_3, ..., A_n\}$ we see that

$$\boxed{\begin{array}{l} P(A_1 \cup A_2 \cup A_3 \cup ... \cup A_n) = \\ \sum_{k=1}^{n} P(A_k) - \sum_{j \neq k} P(A_j \cap A_k) + \sum_{i \neq j \neq k} P(A_i \cap A_j \cap A_k) - ... \end{array}} \tag{79}$$

Appendix B Euler's Number

B.1 Compound Interest

Euler's number can be defined as

$$\boxed{e = \lim_{n \to \infty} (1 + \frac{1}{n})^n} \tag{80}$$

It was originally discovered in the context of calculating continuous interest. Suppose a bank pays you 5% interest once per year on your savings of x dollars. How much money will you have in the account after one year? You'll have

$$x + 0.05x \; = \; x(1 + 0.05) \; = \; 1.05x$$

How much money will you have after 2 years? You'll have

$$1.05x + 0.05(1.05x) \; = \; 1.05x(1 + 0.05) \; = \; 1.05x(1.05) \; = x(1.05)^2$$

In general, if you're payed interest on an amount x, at a rate r, over p periods, the resulting amount of money in the account will be

$$\text{my monies } = x(1 + r)^p$$

What if instead of receiving interest once per year, you're payed monthly? Then the amount of money after one year will be

$$\text{my monies } = x(1 + \frac{r}{12})^{12}$$

This is because instead of your account growing to $x(1 + r)$ dollars at the end of the year, your account is growing to $x(1 + \frac{r}{12})$ dollars every month. What if interest were paid every day? Then at the end of the year you'd have

$$\text{my monies } = x(1 + \frac{r}{365})^{365}$$

Hopefully, you can see where this is going. What is the maximum amount of money you could make on interest if the you were paid with infinite granularity?

$$\boxed{\text{my monies } = \lim_{n \to \infty} x(1 + \frac{r}{n})^n} \tag{81}$$

Notice that Euler's number (Equation 80) is the same as Equation 81 with $r = x = 1$.

B.2 Euler's Function and Infinite Sum

Define Euler's function as

$$\boxed{e(x) = \lim_{n\to\infty}(1+\frac{x}{n})^n} \tag{82}$$

What happens if we try to calculate $(1+\frac{x}{n})^n$? In general, in order calculate $(a+b)^n$ we use the binomial theorem.* The binomial theorm states that

$$\boxed{(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n-1}ab^{n-1} + b^n}$$

Applying the binomial theorem to Euler's function we find that

$$e(x) = \lim_{n\to\infty}(1+\frac{x}{n})^n =$$

$$\lim_{n\to\infty} 1 + \binom{n}{1}\frac{x}{n} + \binom{n}{2}\frac{x^2}{n^2} + \binom{n}{3}\frac{x^3}{n^3} + ... = \tag{83}$$

$$\lim_{n\to\infty} 1 + \frac{n}{1!}\frac{x}{n} + \frac{n(n-1)}{2!}\frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{x^3}{n^3} + ...$$

To simplify this sum, we need to figure out the limit of each term as $n \to \infty$

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)...(n-k+1)}{k!}\frac{x^k}{n^k} \tag{84}$$

Notice that

$$n(n-1)(n-2)...(n-k+1) = n^k + c_1 n^{k-1} + c_2 n^{k-2} + ... + c_{k-1} n$$

for some constants $c_1, c_2, c_3, ...$ that we don't need to calculate for the purposes of figuring out the limit.

*Refer to Append C for an explanation of the binomial theorem.

Substituting this expression back into equation 84:

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)...(n-k+1)}{k!}\frac{x^k}{n^k}$$

$$= \lim_{n\to\infty} \frac{n^k + c_1 n^{k-1} + c_2 n^{k-2} + ... + c_{k-1} n}{k!}\frac{x^k}{n^k}$$

$$= \lim_{n\to\infty} \left[\frac{n^k}{n^k} + \frac{c_1 n^{k-1}}{n^k} + \frac{c_2 n^{k-2}}{n^k} + ... + \frac{c_{k-1} n}{n^k}\right](\frac{x^k}{k!})$$

As $n \to \infty$, all of the terms but the n^k term go to 0

$$= \lim_{n\to\infty} [\frac{n^k}{n^k}](\frac{x^k}{k!}) = \lim_{n\to\infty} (\frac{x^k}{k!}) = \frac{x^k}{k!}$$

Therefore

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)...(n-k+1)}{k!}\frac{x^k}{n^k} = \frac{x^k}{k!} \tag{85}$$

Substituting equation 85 back into Euler's function (equation 83), we get

$$e(x) = \lim_{n\to\infty} 1 + \frac{n}{1!}\frac{x}{n} + \frac{n(n-1)}{2!}\frac{x^2}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{x^3}{n^3} + ...$$
$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

Therefore Euler's function is

$$\boxed{e(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...} \tag{86}$$

We've seen that $e(1) = e = e^1$, but how does $e(x)$ relate to e^x? Might they be the same function?*

B.3 Deriving e^x

Here's how we can show $e(x) = e^x$. We need to use some calculus to do it. I thought I could avoid all calculus aside from limit calculations, but right here at the end, I'm at a loss as to how to avoid it.

*Hint: Yes

So let's try to calculate the derivative* of e^x.

$$\begin{aligned}
\frac{d}{dx}e^x &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\
&= \lim_{h\to 0}\frac{e^x(e^h-1)}{h}\\
&= e^x\lim_{h\to 0}\frac{e^h-1}{h}\\
&\textbf{Substite the formula for } e \textbf{ on the right}\\
&= e^x\lim_{h\to 0}\frac{(\lim_{n\to\infty}(1+\frac{1}{n})^n)^h-1}{h}\\
&\textbf{Let } \boldsymbol{n=\frac{1}{h}}\\
&= e^x\lim_{h\to 0}\frac{(\lim_{h\to 0}(1+\frac{1}{\frac{1}{h}})^{\frac{1}{h}})^h-1}{h}\\
&= e^x\lim_{h\to 0}\frac{((1+\frac{1}{\frac{1}{h}})^{\frac{1}{h}})^h-1}{h}\\
&= e^x\lim_{h\to 0}\frac{(1+\frac{1}{\frac{1}{h}})-1}{h}\\
&= e^x\lim_{h\to 0}\frac{(1+h)-1}{h}\\
&= e^x\lim_{h\to 0}\frac{h}{h} = e^x\lim_{h\to 0}1\\
&= e^x
\end{aligned}$$

The derivative of e^x is e^x. This is actually one of the most useful properties of e^x.†

$$\boxed{\frac{d}{dx}e^x = e^x} \tag{87}$$

Now let's examine the Taylor series expansion‡ of e^x. The taylor series expansion of a function $f(x)$ is

$$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}(0) + ... \tag{88}$$

*The derivative of a function is another function which represents the slope (or rate of change) of that function

†But I'm not going to write an appendix about it.

‡The Taylor series of a function expresses the function as an infinite polynomial. The idea is that if the polynomial and all of its derivatives are equal to the function it's approximating at a particular point, then it will be equivalent to that function. Note that $f'(x)$ is another way to express the derivative of $f(x)$. Similarly, $f''(x)$ is the second derivative, etc...

Since the derivative of e^x is e^x, then its second derivative is also e^x, and, in fact, all of its derivatives are e^x. Therefore the Taylor series for e^x is

$$e^x = e^0 + \frac{e^0 x}{1!} + \frac{e^0 x^2}{2!} + \frac{e^0 x^3}{3!} + ... = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \tag{89}$$

Comparing equation 89 with equation 86 we see that

$$\boxed{e(x) = e^x} \tag{90}$$

Appendix C The Binomial Theorem

The binomial theorem states:

$$\boxed{\begin{array}{l}(a+b)^n = \\ a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{n-1}ab^{n-1} + b^n\end{array}} \tag{91}$$

The reasoning behind the theorem is that $(a+b)^n$ has n copies of $(a+b)$.

$$(a+b)^n = \overbrace{(a+b)(a+b)...(a+b)}^{n \text{ copies}} \tag{92}$$

Think of the a's and b's as $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$. To multiply the binomials, we need to find the all different ways that we can combine the a_i and b_i terms.

The choose function* $\binom{n}{m}$ calculates the number of ways you can pick m elements from a set of n element.

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \tag{93}$$

There's only one way that we can pick all the a_i's. Simply choose the a_i from each term.

There are n ways that we can pick $n-1$ a_i's. Pick all the a_i's but one. You can think of this as picking one of the b_i's – the term where you won't pick the a_i.

In general there are $\binom{n}{m}$ ways that we can pick m a's from the n binomials.

*Also (not coincidentally) known as the binomial coefficient