Fraction Counting Problem

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1 Problem

We are all familiar with counting in integers. If given an integer, N, it's relatively easy to write an algorithm which prints out the integers $1..N$ in order. A similar problem exists in fractions. Given a number, N, find an algorithm which prints out, in order and in reduced form, all of the fractions between 0 and 1 with denominator less than or equal to N.

For example, with $N = 5$ the algorithm should output: $\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$.

2 Background

2.1 Euclidean Algorithm

Given two integers a_0 and a_1, $0 < a_1 \leq a_0$, consider the following sequence of equations:

$a_0 = m_1 a_1 + a_2,\ 0 < a_2 < a_1$
$a_1 = m_2 a_2 + a_3,\ 0 < a_3 < a_2$
.
.
.
$a_{n-2} = m_{n-1} a_{n-1} + a_n,\ 0 < a_n < a_{n-1}$
$a_{n-1} = m_n a_n + a_{n+1},\ a_{n+1} = 0.$

Here we have $m_i = a_{i-1}$ div a_i and $a_i = a_{i-2}$ mod a_{i-1} where div and mod represent integer division and remainder operations. This set of equations represents Euclid's algorithm which is used to find the greatest common divisor (GCD) of two numbers. Specifically, $a_n = GCD(a_0, a_1)$. I'll prove the validity of this statement and some additional useful facts in the next subsections.

2.2 Common Divisors of a_0 and a_1

Claim 1. If $a_0 = da'_0$ and $a_1 = da'_1$ then $a_i = da'_i$ for all i.

What this says is that if a_0 and a_1 are divisible by d, then all of the a_i's are divisible by d.

Proof.

$$\text{Base Case: } a_0 = da'_0 \text{ and } a_1 = da'_1. \tag{1}$$

$$\text{Now assume that } a_{i-1} = da'_{i-1} \text{ and } a_i = da'_i. \tag{2}$$

$$\text{By the Euclidean algorithm, } a_{i-1} = m_i a_i + a_{i+1} \tag{3}$$

$$\Rightarrow a_{i+1} = a_{i-1} - m_i a_i \tag{4}$$

$$(2) \wedge (4) \Rightarrow a_{i+1} = da'_{i-1} - m_i da'_i = d(a'_{i-1} - m_i a'_i) \tag{5}$$

$$\Rightarrow a_{i+1} = da'_{i+1} \tag{6}$$

So by induction, the claim is proven.

Claim 2. If $a_i = da'_i$ and $a'_{i+1} = da'_{i+1}$ then $a_0 = da'_0$ and $a_1 = da'_1$.

What this says is that if a number, d, divides two consecutive a_i's, then it divides a_0 and a_1 as well.

Proof.

$$\text{Assume that } a_i = da'_i \text{ and } a'_{i+1} = da'_{i+1}. \tag{7}$$

$$\text{By the Euclidean algorithm, } a_{i-1} = m_i a_i + a_{i+1} \tag{8}$$

$$(7) \wedge (8) \Rightarrow a_{i-1} = m_i da'_i + da'_{i+1} = d(m_i a'_i + a'_{i+1}) \tag{9}$$

$$\Rightarrow a_{i-1} = da'_{i-1} \tag{10}$$

So by induction, the claim is proven.

2.3 Greatest Common Divisor of a_0 and a_1

Claim 3. There exists an n such that $a_n \neq 0$ and $a_{n+1} = 0$.

What this says is that the Euclidean algorithm is guaranteed to terminate and therefore guranteed to find the GCD of a_0 and a_1.

Proof.

$$\text{By the Euclidean algorithm, } 0 \leq a_{i+1} < a_i \text{ for } i \geq 1. \tag{11}$$

$$\Rightarrow 0 \leq a_{i+1} \leq a_i - 1 \tag{12}$$

$$\Rightarrow 0 \leq a_{i+1} \leq a_0 - i \tag{13}$$

So after at most a_0 steps, a_i will equal 0.

Claim4. a_n is the Greatest Common Divisor of a_0 and a_1.

What this means is that $a_0 = a_n a'_0$ and $a_1 = a_n a'_1$ and there is no number d, such that $d > a_n$ and $a_0 = da''_0$ and $a_1 = da''_1$.

Proof.

$$\text{By the Euclidean algorithm, } a_{n-1} = m_n a_n + a_{n+1},\ a_{n+1} = 0. \tag{14}$$

$$\Rightarrow a_{n+1} = 0 \cdot a_n \text{ and } a_n = 1 \cdot a_n \tag{15}$$

$$\text{(By Claim 2) } \Rightarrow a_0 = a_n a'_0 \text{ and } a_1 = a_n a'_1. \tag{16}$$

So a_n is *a* divisor of a_0 and a_1.

Now assume that $\exists d$ such that $d > a_n$ and $a_0 = da'_0$ and $a_1 = da'_1$.

$$\text{(By Claim 1) } \Rightarrow a_n = da'_n \tag{17}$$

$$d > a_n \Rightarrow a_n = a'_n = 0 \tag{18}$$

which contradicts the Euclidean algorithm's statement that $0 < a_n < a_{n-1}$. So, a_n must be the GCD of a_0 and a_1.

2.4 Linear Integer Equations

Claim 5. Given two integers a_0 and a_1, $0 < a_1 \leq a_0$, the following equation is soluble for integers x and y:

$$a_0 x - a_1 y = GCD(a_0, a_1) \tag{19}$$

Proof.

$$\text{Base Case: } a_0 = a_0 \cdot 1 - a_1 \cdot 0,\ a_1 = a_0 \cdot 0 - a_1 \cdot -1. \tag{20}$$

$$\text{Now assume that } a_{i-1} = a_0 x_{i-1} - a_1 y_{i-1},\ a_i = a_0 x_i - a_1 y_i. \tag{21}$$

$$\text{By the Euclidean Algorithm, } a_{i-1} = m_i a_i + a_{i+1} \tag{22}$$

$$\Rightarrow a_{i+1} = a_{i-1} - m_i a_i \tag{23}$$

$$(21) \wedge (23) \Rightarrow a_{i+1} = a_0 x_{i-1} - a_1 y_{i-1} - m_i(a_0 x_i - a_1 y_i) \tag{24}$$

$$\Rightarrow a_{i+1} = a_0(x_{i-1} - m_i x_i) - a_1(y_{i-1} - m_i y_i) \tag{25}$$

So by induction, $a_i = a_0 x_i - a_1 y_i$ for all i. Indeed, $a_n = a_0 x_n - a_1 y_n$. But by Claim 4, $a_n = GCD(a_0, a_1)$.

2.5 Fundamental Theorem of Arithmetic

Claim 6. Given positive integers, a, b, and c such that $bc = aa'$ and $GCD(a, b) = 1$, we have $c = ac'$.

What this says is that if a divides bc and a and b have no common divisors, then a must divide c.

Proof.

$$\text{By Claim 5, we can find x, y such that } ax - by = GCD(a, b) = 1 \quad (26)$$

$$\Rightarrow c(ax - by) = c \quad (27)$$

$$\Rightarrow acx - bcy = c \quad (28)$$

$$\Rightarrow acx - aa'y = c \quad (29)$$

$$\Rightarrow c = a(cx - a'y) = ac' \quad (30)$$

Claim 6 is the basis for the Fundamental Theorem of Arithmetic which states that any integer can be uniquely factored into a product of prime numbers (i.e. there is a unique prime number representation for every integer.)

It's obvious that an integer *can* be factored into a product of primes. One simply has to continue extracting factors from the integer and factors from the factors until all that's left is primes.

However, the Fundamental Theorem of Arithmetic states that given an integer $a = p_1 p_2 \cdots p_n$ where $p_1, p_2, ..., p_n$ are prime numbers, there is no other representation, $a = q_1 q_2 \cdots q_k$ where $q_1, q_2, ..., q_k$ are prime numbers and the q's and k differ from the p's and n (except, of course, for simple rearrangement of the terms.)

The Fundamental Theorem of Arithmetic, itself, is not necessary for solving the fraction counting problem, so I will only outline the proof.

Suppose that $a = p_1 p_2 \cdots p_n$ where $p_1, p_2, ..., p_n$ are prime numbers and $a = q_1 q_2 \cdots q_k$ where $q_1, q_2, ..., q_k$ are prime numbers.

Then we have that $p_1 p_2 \cdots p_n = q_1 q_2 \cdots q_k$. By Claim 6, p_1 must divide one of the q_i's (in fact, it must divide it exactly since these are prime numbers and thus have no factors in common except themselves.)

Cancel out p_1 and q_i and repeat the process for $p_2, p_3, ..., p_n$ in order to discover that the terms match exactly.

2.6 Complete Solutions to Certain Linear Integer Equations

Claim 7. Given two integers a and b such that $GCD(a, b) = 1$, the following equation is soluble for integers x' and y':

$$ax' - by' = n \tag{31}$$

Proof.

By Claim 6, we can find integers r and s such that

$$ar - bs = GCD(a, b) = 1. \tag{32}$$

$$\Rightarrow n(ar - bs) = n \tag{33}$$

$$\Rightarrow a(nr) - b(ns) = n \tag{34}$$

$$\Rightarrow x' = nr,\ y' = ns \tag{35}$$

Claim 8. All possible solutions to $ax - by = n$ are given by:

$$x = x' + mb,\ y = y' + ma \tag{36}$$

Proof.

$$ax - by = a(x' + mb) - b(y' + ma) \tag{37}$$

$$= ax' + amb - by' - bma \tag{38}$$

$$= ax' - by' = n \tag{39}$$

So x and y satisfy $ax - by = n$.

Now suppose that there exist u and v such that

$$au - bv = n \tag{40}$$

and

$$|\ u - x\ | <\ b. \tag{41}$$

$$\Rightarrow ax - by - (au - bv) = n - n = 0 \tag{42}$$

$$\Rightarrow a(x - u) - b(y - v) = 0 \tag{43}$$

$$\Rightarrow a(x - u) = b(y - v) \tag{44}$$

$$\text{Claim 6 } \wedge\ (44) \Rightarrow y - v = ac' \tag{45}$$

$$\Rightarrow a(x - u) = bac' \tag{46}$$

$$\Rightarrow (x - u) = bc' \tag{47}$$

$$(41)\ \wedge\ (47) \Rightarrow c' = 0 \tag{48}$$

$$\Rightarrow\ x = u \tag{49}$$

$$(43)\ \wedge\ (49) \Rightarrow y = v \tag{50}$$

So $x = x' + mb$ and $y = y' + ma$ are the only solutions to $ax - by = n$.

3 Properties of the Fraction Counting Sequence

3.1 Theorem 1

For any two consecutive fractions in the fraction counting sequence, $\frac{a}{b} < \frac{c}{d}$, $cb - ad = 1$. Or equivalently, $\frac{c}{d} - \frac{a}{b} = \frac{1}{bd}$.

Proof. Given $\frac{a}{b}$, we're interested in finding the closest fraction, $\frac{x}{y}$, which is less than $\frac{a}{b}$ and satisfies the constraints of the problem, namely $y \leq N$ and $GCD(x, y) = 1$.

I claim that solving

$$ay - bx = 1 \tag{51}$$

$$y \leq N \text{ and } y + b > N \tag{52}$$

for x and y gives the desired closest fraction.

By Claim 5, $ay' - bx' = 1$ is soluble since $GCD(a, b) = 1$.

Also, from Claim 8, we have $ay - bx = 1$ where $y = y' + mb$ and $x = x' + ma$ so we can find an appropriate m such that $y = y' + mb$, $y \leq N$, and $y + b > N$.

Now assume that there exist u and v such that

$$v \leq N, \tag{53}$$

$$GCD(u, v) = 1, \tag{54}$$

$$\frac{x}{y} < \frac{u}{v} < \frac{a}{b}, \tag{55}$$

$$av - bu = n, \; n > 1. \tag{56}$$

Using these assumptions we find

$$(55) \Rightarrow \frac{a}{b} - \frac{u}{v} < \frac{a}{b} - \frac{x}{y} \tag{57}$$

$$(56) \Rightarrow \frac{a}{b} - \frac{u}{v} = \frac{n}{bv} \tag{58}$$

$$(57) \;\wedge\; (58) \Rightarrow \frac{n}{bv} < \frac{1}{by} \tag{59}$$

$$\Rightarrow n < \frac{v}{y} \tag{60}$$

By Claim 8, we have

$$u = nx + ka \text{ and } v = ny + kb \text{ for some integer k.} \tag{61}$$

Now we examine 3 cases: $k < 0$, $k = 0$, and $k > 0$.

Case 1: $k < 0$

$$k < 0 \ \wedge \ (61) \Rightarrow v < ny \tag{62}$$

$$\Rightarrow n \geq \frac{v}{y} \tag{63}$$

which contradicts (60).

Case 2: $k = 0$

$$k = 0 \ \wedge \ (61) \Rightarrow u = nx \text{ and } v = ny \tag{64}$$

$$\Rightarrow GCD(u, v) \geq n \tag{65}$$

which contradicts (54) since $n > 1$.

Case 3: $k > 0$

$$k > 0 \ \wedge \ (61) \Rightarrow v \geq ny + b \tag{66}$$

$$(52) \ \wedge \ (66) \Rightarrow v > N. \tag{67}$$

which contradicts (53).

So there are no u and v which meet the constraints of the problem such that $\frac{x}{y} < \frac{u}{v} < \frac{a}{b}$ and $av - bu > 1$. This completes the proof.

3.2 Theorem 2

For any three consecutive fractions, $\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$, in the fraction counting sequence the following (amazing) fact holds:

$$\frac{c}{d} = \frac{a+e}{b+f} \tag{68}$$

This fact is attributed to John Farey who discovered it in 1816 after it had escaped the notice of such great number theorists as Euler and Fermat. For this reason the fraction counting series is usually referred to as the Farey series.

Proof. Using Theorem 1, we obtain the following two equations:

$$bc - ad = 1, \tag{69}$$

$$de - cf = 1. \tag{70}$$

Solving for c, we get

$$(69) \Rightarrow d = \frac{bc-1}{a} \tag{71}$$

$$(70) \Rightarrow d = \frac{cf+1}{e} \tag{72}$$

$$(71) \ \wedge \ (72) \Rightarrow \frac{bc-1}{a} = \frac{cf+1}{e} \tag{73}$$

$$\Rightarrow \frac{bce-e}{ae} = \frac{acf+a}{ae} \tag{74}$$

$$\Rightarrow bce - e = acf + a \tag{75}$$

$$\Rightarrow c = \frac{a+e}{be-af} \tag{76}$$

Solving for d, we get

$$(69) \Rightarrow c = \frac{ad+1}{b} \tag{77}$$

$$(70) \Rightarrow c = \frac{de-1}{f} \tag{78}$$

$$(77) \ \wedge \ (78) \Rightarrow \frac{ad+1}{b} = \frac{de-1}{f} \tag{79}$$

$$\Rightarrow \frac{adf+f}{bf} = \frac{bde-b}{bf} \tag{80}$$

$$\Rightarrow adf + f = bde - b \tag{81}$$

$$\Rightarrow d = \frac{b+f}{be-af} \tag{82}$$

Finally, using (76) and (82) we get,

$$\frac{c}{d} = \frac{\frac{a+e}{be-af}}{\frac{b+f}{be-af}} = \frac{a+e}{b+f}. \tag{83}$$

4 Solution

4.1 Determining the Next Fraction in the Sequence

Suppose we know two consecutive fractions, $\frac{a}{b} < \frac{c}{d}$, in the fraction counting sequence. The goal is to find the next fraction, $\frac{e}{f}$, in the sequence.

Theorem 3 If $\frac{a}{b}$ and $\frac{c}{d}$ are two consecutive fractions in the fraction counting sequence such that $\frac{a}{b} < \frac{c}{d}$, then the next fraction, $\frac{e}{f}$, in the sequence can be determined as follows:

$$e = xc - a,\ \ f = xd - b \tag{84}$$

$$x = (N + b) \text{ div } d \tag{85}$$

Proof.

$$\text{By Theorem 2, } \frac{c}{d} = \frac{a+e}{b+f} \tag{86}$$

$$\Rightarrow c(b+f) = d(a+e) \tag{87}$$

$$\text{By the constraints of the problem, } GCD(c,d) = 1. \tag{88}$$

$$(88)\ \wedge\ \text{Claim 6} \Rightarrow b + f = xd \tag{89}$$

$$(87) \wedge\ (89) \Rightarrow cxd = d(a+e) \Rightarrow a + e = xc \tag{90}$$

$$(89)\ \wedge\ (90) \Rightarrow e = xc - a,\ \ f = xd - b \tag{91}$$

which proves (84).

In order to prove (85), consider the function

$$f(x) = \frac{xc - a}{xd - b} \tag{92}$$

Differentiating $f(x)$, we find

$$f'(x) = \frac{c(xd-b) - d(xc-a)}{(xd-b)^2} = \frac{ad - bc}{(xd-b)^2} \tag{93}$$

$$\frac{a}{b} < \frac{c}{d} \Rightarrow ad < bc \Rightarrow ad - bc < 0 \tag{94}$$

$$(93)\ \wedge\ (94) \Rightarrow f'(x) < 0 \text{ for all x for which } f'(x) \text{ is defined.} \tag{95}$$

Therefore, $f(x)$ is monotonically decreasing with x. Also,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{xc-a}{xd-b} = \lim_{x\to\infty} \frac{c - \frac{a}{x}}{d - \frac{b}{x}} = \frac{c}{d} \tag{96}$$

And since $\frac{e}{f} = f(x)$ for some x, we want to pick the largest possible x within the constraints of the problem in order to get $\frac{e}{f}$ as close as possible to $\frac{c}{d}$. If we didn't pick such an x then we could find a larger x which would give us a smaller fraction thus contradicting the problem statement since we wouldn't have all fractions in order.

The constraints of the problem state that we can make x as big as we want so long as $xd - b \leq N$.

$$\Rightarrow xd \leq N + b \tag{97}$$

$$\Rightarrow x \leq \frac{N + b}{d} \tag{98}$$

Since x must be an integer, we get

$$x = (N + b) \text{ div } d \tag{99}$$

which proves (85).

4.2 The Fraction Counting Algorithm

```
void FractionCount(int N) {
    int a,b,c,d,e,f,x;

    a = 0; b = 1;
    c = 1; d = N;

    while(c < d)
    {
        printf("%d/%d",c,d);

        x = (N + b)/d;
        e = x*c - a;
        f = x*d - b;

        a = c; b = d;
        c = e; d = f;

        if(c < d) printf(", ");
    }
    printf("\n");
}
```

The proof of the algorithm follows almost directly from Theorem 3. There's a small catch, though, because we start out with $\frac{a}{b} = \frac{0}{1}$, and we only defined the Euclidean Algorithm in terms of $0 < a \leq b$. However, we can define $GCD(0, n) = n$ for $n > 0$. It's a simple exercise to show that Claim 5 still holds in this case which implies that Claims 6, 7, and 8 hold as well.