Coconut Divisibility Problem

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1 Problem

N boys go camping one weekend. While in the wilderness, the boys collect a number of coconuts. They take the coconuts back to their cabin and agree to divide them up equally before they leave the next morning. That night, one of the boys wakes up (from excitement) and decides to gather his share of coconuts right that moment. However, when he counts the coconuts he finds that the number is not divisible by n, but by giving one coconut to the pet monkey, the remaining amount *is* divisible by n, so the boy gives one coconut to the monkey and takes $\frac{1}{n}$ of the rest for himself.

Later that night another boy wakes up (also from excitement) and decides to gather his share of coconuts right that moment (not realizing that the other boy has already taken his share.) When he counts the number of coconuts he finds that if he gives one to the pet monkey then the remaining amount will be divisible by n, so the boy gives one coconut to the monkey (who has already disposed of the previous one) and takes $\frac{1}{n}$ of the rest for himself.

Similarly, all of the $n-2$ other boys awake one by one in the middle of the night in order to get their share of coconuts, and all of them have to give one coconut to the monkey before taking $\frac{1}{n}$ of the rest.

In the morning, as the boys are leaving the cabin, they notice that there are still some coconuts left. Since each of them has taken their share, they're not really sure why there should be any left. Not wanting to think about it too hard, though, they just decide to split the remaining coconuts evenly since the remaining amount is exactly divisible by n.

The question, then, is how many coconuts were there to begin with? Give the minimum possible value.

2 Solution

Let a equal the starting number of coconuts and nb equal the number of coconuts left in the morning. Also notice that $c - \frac{c}{n} = (c)(\frac{n-1}{n})$ so instead of constantly subtracting $\frac{1}{n}$ of the remaining amount from the remaining amount we can simply multiply the remaining amount by $\frac{n-1}{n}$. Now let $x = \frac{n-1}{n}$. This gives the following equation:

$$(\cdots(((a-1)x-1)x-1)x\cdots-1)x = nb \tag{1}$$

where we subtract 1 and then multiply by x, n times.

$$\Rightarrow ax^n - (x + x^2 + x^3 + \cdots + x^n) = nb \tag{2}$$

Then by adding and subtracting 1 on the left we get:

$$ax^n - (1 + x + x^2 + x^3 + \cdots + x^n) + 1 = nb \tag{3}$$

$$\Rightarrow ax^n - \frac{x^{n+1}-1}{x-1} + 1 = nb \tag{4}$$

$$\Rightarrow ax^n = \frac{x^{n+1}-1}{x-1} - 1 + nb \tag{5}$$

Now substituting back $\frac{n-1}{n}$ for x we get:

$$a(\frac{n-1}{n})^n = \frac{(\frac{n-1}{n})^{n+1}-1}{\frac{n-1}{n}-1} - 1 + nb \tag{6}$$

And since $\frac{n-1}{n} - 1 = -\frac{1}{n}$, we have:

$$a(\frac{n-1}{n})^n = -n((\frac{n-1}{n})^{n+1} - 1) - 1 + nb \tag{7}$$

$$\Rightarrow a(\frac{n-1}{n})^n = n - n(\frac{n-1}{n})^{n+1} - 1 + nb \tag{8}$$

$$\Rightarrow a(\frac{n-1}{n})^n = -n(\frac{n-1}{n})^{n+1} - 1 + n(b+1) \tag{9}$$

Now multiplyling by n^n on both sides we get:

$$a(n-1)^n = -(n-1)^{n+1} - n^n + n^{n+1}(b+1) \tag{10}$$

Now suppose that we had a pair (a, b) which satisfied (10). Notice then that $(a + tn^{n+1}, b + t(n-1)^n)$ also satisfies the equation since the extra $tn^{n+1}(n-1)^n$ cancels out on both sides. Therefore, valid values of a occur at intervals of tn^{n+1}. From now on I'll just write the equation like this:

$$a(n-1)^n \equiv -(n-1)^{n+1} - n^n \pmod{n^{n+1}} \tag{11}$$

This is equivalent to writing the exact equation:

$$a(n-1)^n = -(n-1)^{n+1} - n^n + tn^{n+1} \tag{12}$$

where t is just some integer. This is ok, because in the end if we're able to get an equation like this:

$$a \equiv c \pmod{n^{n+1}} \tag{13}$$

then this really means

$$a = c + tn^{n+1} \tag{14}$$

for some integer t. But remember that valid values of a repeat every n^{n+1} so another valid value would be $a - tn^{n+1}$ which simply equals c.

Also notice the following properties of modular arithmetic. Assume $a = b + ct$:

$$a = b + ct \Leftrightarrow a \equiv b \pmod{c} \tag{15}$$

$$ax = bx + ctx \Leftrightarrow ax \equiv bx \pmod{c} \tag{16}$$

$$a + x = b + x + ct \Leftrightarrow a + x \equiv b + x \pmod{c} \tag{17}$$

So we can add and multiply with integers in a modular equation just as we do in a normal equation.

Now consider the integer:

$$y = 1 + n + n^2 + \cdots + n^n \tag{18}$$

Notice that:

$$y = \frac{n^{n+1} - 1}{n - 1} \tag{19}$$

$$\Rightarrow y^n = \frac{cn^{n+1} + (-1)^n}{(n-1)^n} \tag{20}$$

where c is some integer.

Now going back to (11) and multiplying by y^n on both sides we get:

$$a(n-1)^n y^n \equiv -(n-1)^{n+1} y^n - n^n y^n \pmod{n^{n+1}} \tag{21}$$

$$\Rightarrow a(cn^{n+1} + (-1)^n) \equiv -(n-1)(cn^{n+1} + (-1)^n) - (ny)^n \pmod{n^{n+1}} \tag{22}$$

Just lumping the acn^{n+1} and $-(n-1)cn^{n+1}$ terms into the generic tn^{n+1} term, we get:

$$a(-1)^n \equiv -(n-1)(-1)^n - (ny)^n \pmod{n^{n+1}} \tag{23}$$

Then by multiplying by $(-1)^n$ on both sides and noting that $(-1)^{2n} = 1$, we get:

$$a \equiv -(n-1) - (ny)^n(-1)^n \pmod{n^{n+1}} \tag{24}$$

$$\Rightarrow a \equiv (-1)^{n+1}(ny)^n - (n-1) \pmod{n^{n+1}} \tag{25}$$

Using the fact that $y = 1 + n + n^2 + \cdots + n^n$, we have:

$$ny = n + n^2 + n^3 + \cdots + n^{n+1} \tag{26}$$

$$\Rightarrow (ny)^n = (n + n^2 + n^3 + \cdots + n^{n+1})^n \tag{27}$$

Now since every term of this product must be the result of multiplying n terms chosen from $(n + n^2 + n^3 + \cdots + n^{n+1})$, every term of the result must be a multiple of n^{n+1} except for the one in which we pick n n's. So this gives:

$$(ny)^n = n^n + dn^{n+1} \tag{28}$$

for some integer d.

Now plugging the value for $(ny)^n$ back into (25) and lumping the dn^{n+1} with the generic tn^{n+1} term we get:

$$\Rightarrow a \equiv (-1)^{n+1}n^n - (n-1) \pmod{n^{n+1}} \tag{29}$$

$$\Rightarrow a = (-1)^{n+1}n^n - (n-1) + tn^{n+1} \tag{30}$$

Since we can subtract any multiple of n^{n+1} from a and have another valid result, just subtract tn^{n+1} to get:

$$\Rightarrow a = (-1)^{n+1}n^n - (n-1) \tag{31}$$

If n is odd then we have our answer. If n is even, then we must add back one term of n^{n+1} to the right in order to make a positive. So if n is even then we get:

$$a = n^{n+1} - n^n - (n-1) \tag{32}$$

$$\Rightarrow a = n^n(n-1) - (n-1) \tag{33}$$

Putting the two cases together we get:

$$n = \begin{cases} n^n - (n-1) & n \text{ odd} \\ (n-1)n^n - (n-1) & n \text{ even} \end{cases}$$